

Scientific journal  
**PHYSICAL AND MATHEMATICAL EDUCATION**  
 Has been issued since 2013.

ISSN 2413-158X (online)  
 ISSN 2413-1571 (print)

Науковий журнал  
**ФІЗИКО-МАТЕМАТИЧНА ОСВІТА**  
 Видається з 2013.



<http://fmo-journal.fizmatsspu.sumy.ua/>

Кузьмич В.І., Кузьмич Л.В., Савченко О.Г. Формування поняття кута засобами метричної геометрії на геометричному матеріалі 9 класу. *Фізико-математична освіта*. 2021. Випуск 3(29). С. 6-12.

Kuz'mich V., Kuzmich L., Savchenko A. Formation of the concept of angle by means of metric geometry on geometric material of 9th grade. *Physical and Mathematical Education*. 2021. Issue 3(29). P. 6-12.

DOI 10.31110/2413-1571-2021-029-3-001  
 UDC 372.851

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## FORMATION OF THE CONCEPT OF ANGLE BY MEANS OF METRIC GEOMETRY ON GEOMETRIC MATERIAL OF 9TH GRADE

### ABSTRACT

The work is to study the possibility of using the means of metric geometry for the formation of basic geometric concepts in the study of geometry in secondary education. The use of metric geometry opens the way for students to get acquainted with the elements of non-Euclidean geometries on both intuitive and axiomatic levels. In the paper, an alternative definition of the rectilinear placement of points is given based on the definition of the numerical characteristic of the angle formed by the three points of the metric space. With the help of numerical characteristics, it is easy to obtain definitions of right and straight angles. The material presented in this paper can be used in geometry lessons starting from the 9th year and in extracurricular activities with students studying in classes with in-depth mathematics learning.

**Formulation of the problem.** The material of this work relates mainly to the teaching of mathematics in classes with in-depth mathematics learning. The current mathematical education state raises the question of the need to acquaint students with the basic concepts and facts of non-Euclidean geometries. Doing this directly by referring to the actual material about such geometries is quite difficult because of the significant level of its formalization. To address this issue, the authors offer to use the means of metric geometry as the closest to the school course of geometry. It is offered to begin this work with the formation of generalized basic geometric concepts and objects, such as a point, an angle, and a rectilinear placement of points.

**Materials and methods.** The results are based on the analysis of existing textbooks in mathematics for classes with in-depth study, textbooks on geometry and mathematical analysis of higher education institutions, scientific publications and tested when reading the relevant special course for students majoring in "014.04 Secondary Education (Mathematics)" master's degree.

**Results.** Based on the proposed definition of the angle as an ordered triple of points, analogs of classical geometric relations are obtained. These analogues allow the demonstration of elements of non-Euclidean geometries by means of elementary geometry.

**Conclusions.** The analytical apparatus of metric geometry makes it possible to form a generalized understanding of basic geometric concepts, such as point, angle, distance between points, rectilinear placement of points.

**KEY WORDS:** point, angle, distance, metric, metric space.

### INTRODUCTION

**The problem formulation.** Geometry textbooks use two different definitions of a plane angle: as a set of two rays with a common origin, and as part of a plane limited by these rays. In this paper, it is offered to supplement these definitions of the angle, considering it as an ordered three points. This definition of angle makes it possible to use it to describe the concept of rectilinear placement of points. In this case, it becomes possible to use the conceptual and analytical apparatus of metric geometry to obtain relations of a geometric nature. A small number of simple axioms that can be described in words or written

analytically makes it possible to use metric geometry to form such basic geometric concepts as a point, distance, rectilinear placement of points, an angle in the study of relevant material in the school geometry course. Also it is possible to consider these concepts in a broader sense than it is accepted in existing school textbooks.

**Research relevance.** This work is a logical continuation of the work (Kuz'mich, 2020) and is devoted to the possibility of acquainting 9th grade students with the elements of non-Euclidean geometries based on metric geometry. The method of studying the basic elements of non-Euclidean geometries (in particular, spherical geometry) has been considered by a number of authors. For example, (Lénárt 2020) studied the relationship between the concepts of "point" and "line" in different types of geometries - spherical, hyperbolic and flat, by constructing an appropriate algebraic system (comparative geometry). The paper (Lénárt & Rybak, 2017) describes the approach to the use of comparative geometry in the school course of mathematics, within the reform of mathematics education in Hungary.

**The aim of the article.** The aim of the research is to demonstrate the means of metric geometry, with the help of which it is possible to form in pupils generalized notions of distance between points, rectilinear placement of points, angle. Such concepts provide an opportunity to acquaint pupils with the elements of non-Euclidean geometries.

## RESEARCH METHODS

The main method of research is to use the concept of metric space to formulate basic geometric concepts. The main results of the study were obtained using three axioms of the distance between the points of the metric space. The paper uses some geometric relationships with which 9th grade students are familiar, and it is shown that these relationships are performed in an arbitrary metric space.

## RESULTS

In this paper we consider several ways to form the concept of a flat angle. It should be noted that this concept is ambiguous, and even in existing school textbooks in mathematics there are several definitions of angle.

In the "Elements" of Euclid, the following definition of angle is given: "A flat angle is the inclination of two lines that meet in a plane with each other, but which are not placed on (one) line" (Mordukhay-Boltovskiy, 1948).

In current mathematics textbooks, when defining an angle, the geometric system of D. Hilbert is followed: "Let  $\alpha$  be an arbitrary plane, and let  $h, k$  be any two different rays in the plane  $\alpha$ , emanating from the point  $O$  and belonging to different lines. We call the system of these two rays  $h, k$  an angle and denote  $\angle(h, k)$  or  $\angle(k, h)$ " (Hilbert, 1923).

In accordance with the definition in the textbook of mathematics for the fifth year the concept of angle is also considered as a set of two rays with a common origin, it is emphasized that in writing the angle an order of letters is important (Merzlyak, Polonsky & Yakir, 2018).

In the geometry textbook for the seventh grade the angle is understood as a part of the plane bounded by two rays with a common origin (Merzlyak, Polonsky & Yakir, 2015).

Each of the concepts of angle above in our opinion can be used depending on the specific conditions of its usage. Namely, the angle can be understood as a system of two rays with a common origin and as a part of the plane bounded by these rays. In addition to these definitions it is offered to consider another definition of an angle, which does not require the concepts of a straight line and a plane. To do this you need to turn to the means of metric geometry.

The basic concepts of metric geometry are the concept of a point as an element of some set  $A$  and the distance  $\rho$  between two points  $x$  and  $y$  of the set  $A$ . The distance  $\rho(x, y)$  must satisfy with three axioms. This set of points is called a metric space and is denoted by  $(A, \rho)$ .

The axioms of distance are quite simple and understandable even for fifth graders. The first one requires the distance between any two different points  $x$  and  $y$  of the set  $A$  be positive ( $\rho(x, y) > 0$ ). The second axiom requires commutative distance: the distance between any points  $x$  and  $y$  of the set  $A$  is the same as between the points  $y$  and  $x$  ( $\rho(x, y) = \rho(y, x)$ ). The third axiom of distance is called "triangle inequality". It requires the distance between any points  $x$  and  $z$  of the set  $A$  be no greater than the sum of the distances from these points to any third point  $y$  of this set ( $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ ). If in the inequality of a triangle the sign of inequality is replaced by the sign of equality ( $\rho(x, z) = \rho(x, y) + \rho(y, z)$ ), in this case it is said that the points  $x, y, z$  are rectilinearly placed in space  $(A, \rho)$ , and the point  $y$  is between the points  $x$  and  $z$  (is internal to the points  $x, y, z$ ).

Consider the concept of angle formed by three different points of the metric space (Kuz'mich, 2019). In the future, for simplicity, we will consider all points as pairwise different. That is, all distances between points will be considered not equal to zero.

*Definition 1.* Let  $x, y, z$  – be three different arbitrary points of the metric space  $(A, \rho)$ . The ordered triple  $(x, y, z)$  of these points will be called the angle with the vertex at the point  $y$ , and will be denoted as:  $\angle(x, y, z)$ . Pairs of points  $(x, y)$  i  $(y, z)$  in this case will be called the sides of the angle.

This definition of angle is a logical complement to the two definitions mentioned above. That is, depending on the need, the angle can be understood as a characteristic of the relative position of three points relative to one of them, and as a set of rays with a common origin, and as part of the plane bounded by these rays. In this case, the rays can form a straight line, then this angle is called expanded (Merzlyak, Polonsky & Yakir, 2018).

The numerical characteristic of the angle (angle measurement) is based on the uniform division of the expanded angle into 180 equal parts (degrees) (Merzlyak, Polonsky & Yakir, 2018). Based on this division the angles are compared with each other, indicating that equal angles have equal degrees. An angle whose degree is less than  $90^\circ$  is called acute, and an angle whose degree is greater than  $90^\circ$  but less than  $180^\circ$  is called obtuse (Merzlyak, Polonsky & Yakir, 2018). The equality of degree measures of two angles is equivalent to the equality of these angles (Merzlyak, Polonsky & Yakir, 2015).

So finding the degree of the angle is based on the process of dividing the expanded angle at the level of the part by means of geometry. Such a process will necessarily raise the question of its possibility.

If we take a circle as the basis of a numerical measure of an angle, we obtain a radian measure of an angle. In this case, the angle of one radian is called the central angle of the circle, based on an arc whose length is equal to the radius of the circle (Merzlyak, Nomirovsky, Polonsky & Yakir, 2018). In this case, it is necessary to use the concept of a circle to measure the angle, and again the question arises about the possibility of measuring, for example, a right angle.

In the case of defining an angle as an ordered triple of points in a metric space, it is easier to enter its numerical characteristic. You can only use the distances between each pair of these points. This definition can be given on the basis of the cosine theorem, which connects the cosine of the angle of a triangle with the lengths of its sides (Merzlyak, Polonsky & Yakir, 2017).

*Definition 2.* Let  $x, y, z$  – be three different arbitrary points of the metric space  $(A, \rho)$ . The characteristic of the angle  $\angle(x, y, z)$  (the angular characteristic) we call the real number  $\varphi(x, y, z)$ , which can be found as:

$$\varphi(x, y, z) = \frac{\rho^2(x, y) + \rho^2(y, z) - \rho^2(x, z)}{2\rho(x, y)\rho(y, z)} \tag{1}$$

From this definition it is easy to get the definition of the following angles. The “right angle” is called the angle  $\angle(x, y, z)$  for which the equality  $\varphi(x, y, z) = 0$  is right; “expanded angle” - the angle for which the equality  $\varphi(x, y, z) = -1$  is right; “zero angle” - the angle for which the equality  $\varphi(x, y, z) = 1$  is right.

From Definition 2 you can get an estimate for the values of the angular characteristic.

*Theorem 1.* The characteristic of the angle  $\angle(x, y, z)$  satisfies the double inequality:

$$-1 \leq \varphi(x, y, z) \leq 1.$$

Indeed, using the axioms of the distance between the points of the metric space, we will have:

$$\begin{aligned} -1 \leq \varphi(x, y, z) &= \frac{\rho^2(x, y) + \rho^2(y, z) - \rho^2(x, z)}{2\rho(x, y)\rho(y, z)} \leq 1; \\ -2\rho(x, y)\rho(y, z) &\leq \rho^2(x, y) + \rho^2(y, z) - \rho^2(x, z) \leq 2\rho(x, y)\rho(y, z); \\ \begin{cases} \rho^2(x, y) + \rho^2(y, z) - \rho^2(x, z) &\leq 2\rho(x, y)\rho(y, z), \\ -2\rho(x, y)\rho(y, z) &\leq \rho^2(x, y) + \rho^2(y, z) - \rho^2(x, z); \end{cases} \\ \begin{cases} (\rho(x, y) - \rho(y, z))^2 &\leq \rho^2(x, z), \\ (\rho(x, y) + \rho(y, z))^2 &\geq \rho^2(x, z); \end{cases} \end{aligned}$$

Extracting the square root of both parts of the inequalities, we have:

$$\begin{cases} \begin{cases} -\rho(x, z) \leq \rho(x, y) - \rho(y, z) \leq \rho(x, z); \\ \begin{cases} \rho(x, y) + \rho(y, z) \geq \rho(x, z), \\ \rho(x, y) + \rho(y, z) \leq -\rho(x, z); \end{cases} \end{cases} & \begin{cases} \rho(x, y) - \rho(y, z) \leq \rho(x, z), \\ \rho(x, y) - \rho(y, z) \geq -\rho(x, z); \\ \rho(x, y) + \rho(y, z) \geq \rho(x, z); \end{cases} \\ \begin{cases} \rho(x, y) - \rho(y, z) \leq \rho(x, z), \\ \rho(x, y) - \rho(y, z) \geq -\rho(x, z), \\ \rho(x, y) + \rho(y, z) \geq \rho(x, z); \end{cases} & \begin{cases} \rho(x, y) \leq \rho(x, z) + \rho(y, z), \\ \rho(x, y) + \rho(x, z) \geq \rho(y, z), \\ \rho(x, y) + \rho(y, z) \geq \rho(x, z). \end{cases} \end{cases}$$

Each of the inequalities of the obtained system is fulfilled due to the third axiom of the distance between the points of the metric space - "inequalities of the triangle". Thus, the statement of the theorem is valid.

Using Definition 2, we can give an alternative definition of the rectilinear placement of three different points of the metric space (Kuz'mich, 2019).

*Definition 3.* We will say that three different points  $x, y, z$  of the space  $A$  are arranged rectilinear, if for at least one of these points (for example, for the point  $y$ ) the equality holds:

$$\varphi^2(x, y, z) = 1. \tag{2}$$

If we have a set of points, then it is natural to give the following definition of their rectilinear placement (Kuz'mich, 2019).

*Definition 4.* We will say that the set of points of a metric space are rectilinearly placed if any three different points of this set are rectilinearly placed.

Let's show that Definition 3 is equivalent to the classical definition of a rectilinear placement of points. To do this, assume that for three different points  $x, y, z$  of the metric space  $(A, \rho)$  the equality holds:  $\rho(x, z) = \rho(x, y) + \rho(y, z)$ . It means that these points are rectilinearly placed in space  $(A, \rho)$  (Kagan, 1963; Dovgoshei & Dordovskii, 2009; Halushchak, 2016). Subtract both parts of the equality to the square:

$$\rho^2(x, z) = \rho^2(x, y) + 2\rho(x, y)\rho(y, z) + \rho^2(y, z).$$

Substituting this value in the right part of equation (1) we obtain:

$$\begin{aligned} \varphi(x, y, z) &= \frac{\rho^2(x, y) + \rho^2(y, z) - \rho^2(x, z)}{2\rho(x, y)\rho(y, z)} = \\ &= \frac{\rho^2(x, y) + \rho^2(y, z) - (\rho^2(x, y) + 2\rho(x, y)\rho(y, z) + \rho^2(y, z))}{2\rho(x, y)\rho(y, z)} = -1. \end{aligned}$$

Therefore, equality (2) holds, and the points  $x, y, z$  are rectilinearly placed in the sense of Definition 3.

Let now for the points  $x, y, z$  of the space  $(A, \rho)$  the equality (2) from the Definition 3 holds. Then two cases are possible:  $\varphi(x, y, z) = 1$  or  $\varphi(x, y, z) = -1$ .

Consider the first case when the equation is valid:  $\varphi(x, y, z) = 1$ . Then from equality (1) of Definition 2 we obtain the equality:

$$\begin{aligned} \frac{\rho^2(x, y) + \rho^2(y, z) - \rho^2(x, z)}{2\rho(x, y)\rho(y, z)} &= 1; \\ \rho^2(x, y) + \rho^2(y, z) - \rho^2(x, z) &= 2\rho(x, y)\rho(y, z); \\ (\rho(x, y) - \rho(y, z))^2 &= \rho^2(x, z). \end{aligned}$$

The last equality is equivalent to the combination of two equalities:

$$\begin{cases} \rho(x, y) - \rho(y, z) = \rho(x, z), \\ \rho(x, y) - \rho(y, z) = -\rho(x, z); \end{cases}$$

or a set of equations:

$$\begin{cases} \rho(x, y) = \rho(x, z) + \rho(y, z) = \rho(x, z) + \rho(z, y), \\ \rho(y, z) = \rho(x, y) + \rho(x, z) = \rho(y, x) + \rho(x, z). \end{cases}$$

In both cases points  $x, y, z$  are rectilinearly placed in space  $(A, \rho)$ , according to the classical definition (Kagan, 1963). Moreover, in the first case, the point  $z$  lies between the points  $x$  and  $y$ , in the second - the point  $x$  is between  $y$  and  $z$ .

Now let the equation be right:  $\varphi(x, y, z) = -1$ . From equality (1) of Definition 2 we obtain the equality:

$$\frac{\rho^2(x, y) + \rho^2(y, z) - \rho^2(x, z)}{2\rho(x, y)\rho(y, z)} = -1;$$

$$\rho^2(x, y) + \rho^2(y, z) - \rho^2(x, z) = -2\rho(x, y)\rho(y, z);$$

$$(\rho(x, y) + \rho(y, z))^2 = \rho^2(x, z).$$

The last equality is equivalent to the combination of two equalities:

$$\begin{cases} \rho(x, y) + \rho(y, z) = \rho(x, z), \\ \rho(x, y) + \rho(y, z) = -\rho(x, z). \end{cases}$$

From the first equality of the obtained set it follows that the points  $x, y, z$  are rectilinearly placed in space  $(A, \rho)$ , and point  $y$  is between  $x$  and  $z$ . The second equality of the population cannot be satisfied due to the first axiom of distance - distance cannot be negative.

Finally, we obtain that both definitions of the rectilinear placement of the points of the metric space are equivalent to each other.

From equality (1) of Definition 2 we can obtain some well-known classical facts concerning the geometry of Euclid. It is interesting that their geometric meaning is not used. That is, these results are purely analytical in nature, and are the consequences of the properties of the set of real numbers. In particular, from equality (1) it is easy to obtain an analogue of the Pythagorean theorem for a right triangle (Kuz'mich, & Savchenko, 2019).

*Theorem 2.* If for three different points  $x, y, z$  of the metric space  $(A, \rho)$  following equality is right:  $\varphi(x, y, z) = 0$ , then the equality holds:

$$\rho^2(x, z) = \rho^2(x, y) + \rho^2(y, z).$$

To prove this theorem, it suffices to put in equality (1):  $\varphi(x, y, z) = 0$ , and to equate to zero the numerator of the right-hand side of the equality. It is interesting that this result is valid in any metric space and therefore is a consequence of the properties of the set of real numbers.

The following result is also valid in any metric space, that is, does not depend on the method of choosing the distance between points in space (Kuz'mich, & Savchenko, 2019).

*Theorem 3.* For arbitrary three different points  $x, y, z$  of the space  $(A, \rho)$  following equality is right:

$$\rho(x, z) = \rho(x, y)\varphi(y, x, z) + \rho(y, z)\varphi(x, z, y). \tag{3}$$

To prove equality (3) we use equality (1) Definition 2. Transforming the right part of equality (3), we obtain:

$$\begin{aligned} & \rho(x, y)\varphi(y, x, z) + \rho(y, z)\varphi(x, z, y) = \\ & = \rho(x, y) \frac{\rho^2(y, x) + \rho^2(x, z) - \rho^2(y, z)}{2\rho(y, x)\rho(x, z)} + \rho(y, z) \frac{\rho^2(x, z) + \rho^2(z, y) - \rho^2(x, y)}{2\rho(x, z)\rho(z, y)} = \\ & = \frac{\rho^2(y, x) + \rho^2(x, z) - \rho^2(y, z) + \rho^2(x, z) + \rho^2(z, y) - \rho^2(x, y)}{2\rho(x, z)} = \frac{2\rho^2(x, z)}{2\rho(x, z)} = \rho(x, z). \end{aligned}$$

Therefore, equality (3) holds. Note that it is an analogue of the well-known "projection formula" in Euclidean geometry (Ponarin, 2004).

Despite the significant analogy of the obtained results with the corresponding facts of Euclidean geometry, the above definition of the angle allows elements of non-Euclidean geometry. To make sure of this, consider a fairly simple example of a metric space, the points of which are linear functions denoted by the interval  $[0; 1]$ . For the distance between the two functions  $y = f(x)$  and  $y = g(x)$  let's take the number:

$$\rho(f, g) = \max_{x \in [0; 1]} |f(x) - g(x)|. \tag{4}$$

With this choice of distance, the considered set of functions becomes a metric space (Kolmogorov, & Fomin, 1974).

Example 1. Let's consider four functions:  $y_1 = x + 1, y_2 = x, y_3 = x - 2, y_4 = -x$ . On the segment  $[0; 1]$  we find the distances between these functions by formula (4):

$$\begin{aligned} \rho(y_1, y_2) = \rho_{12} = 1; \rho(y_1, y_3) = \rho_{13} = 3; \rho(y_1, y_4) = \rho_{14} = 3; \\ \rho(y_2, y_3) = \rho_{23} = 2; \rho(y_2, y_4) = \rho_{24} = 2; \rho(y_3, y_4) = \rho_{34} = 2. \end{aligned}$$

These results are easy to illustrate on the coordinate plane (Fig. 1).

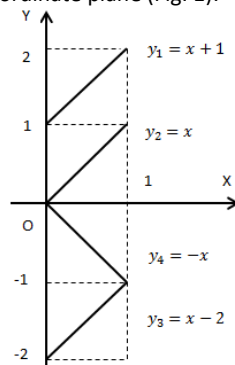


Fig. 1. Graphs of functions  $y_1, y_2, y_3, y_4$

From the obtained equations it follows that the points  $y_1, y_2, y_3$  are rectilinearly placed, because the equality is fulfilled:  $\rho_{13} = \rho_{12} + \rho_{23} = 1 + 2 = 3$ . In this case, the point  $y_2$  lies between the points  $y_1$  and  $y_3$ .

As in the Euclidean geometry, it should be expected that the angle  $\angle(y_1, y_2, y_4)$  and the angle  $\angle(y_3, y_2, y_4)$  will complement each other to the expanded angle. To check this, we find the appropriate angular characteristics. By formula (1) of Definition 2 will have:

$$\varphi(y_1, y_2, y_4) = \varphi_{124} = \frac{\rho_{12}^2 + \rho_{24}^2 - \rho_{14}^2}{2\rho_{12}\rho_{24}} = \frac{1 + 4 - 9}{4} = -1;$$

$$\varphi(y_3, y_2, y_4) = \varphi_{324} = \frac{\rho_{32}^2 + \rho_{24}^2 - \rho_{34}^2}{2\rho_{32}\rho_{24}} = \frac{4 + 4 - 4}{8} = 0,5.$$

Because of Definition 2 one should expect equality:  $\varphi_{124} = -\varphi_{324}$ . However, it is not fulfilled. This indicates that the Definition 1 of angle formed by three points of the metric space allows elements of non-Euclidean geometry. This fact indicates the possibility of using such elements in the school course of mathematics.

## DISCUSSION

The results of the study were reported and discussed at several international scientific and practical conferences, in particular, at the XII International Conference on Mathematics, Science and Technological Education "ICon-MaSTEd 2020" (Kuz'mich & Kuzmich, 2021). These results were tested at Kherson State University. First-year students of the master's level of higher education in the specialty "014.04 Secondary Education (Mathematics)" were taught a special course on the elements of metric geometry. The results of the analysis of the study of this discipline indicate a significant deepening of students' understanding of the basics of geometry, its axiomatics. Almost two hundred years have passed since the first non-Euclidean geometry (Lobachevsky's geometry), however, in the school course of mathematics, even in classes with in-depth study of mathematics, it is still mentioned only historically (Merzlyak, Polonsky & Yakir, 2015). The essence of Lobachevsky's geometry and other non-Euclidean geometries, their interpretations, remain inaccessible to students. The rapid development of metric geometry in our time, the simplicity and accessibility of its axiomatics opens the possibility of acquainting students with the simplest concepts of this geometry, the formation of generalized concepts of point, distance between points, straightness, angle, plane. Such generalized concepts will ensure adequate students' perception of the basic provisions of non-Euclidean geometries.

The study of opportunities for pupils to get acquainted with the elements of non-Euclidean geometries is usually carried out within the framework of non-formal education, using such forms as electives, special math classes and so on. Many dissertation researches are devoted to this topic. In particular, pupils' acquaintance with the elements of Lobachevsky's geometry and other non-Euclidean geometries was studied in detail (Gaybullayev, 1972; Martirosyan, 1973). The development of Euclidean and non-Euclidean spatial representations in high school students was studied while studying the integrative elective course "Introduction to Modern Geometry of the Universe" (Yermak, 1991), as well as the possibility of studying elements of Lobachevsky's geometry at school (Titova, 2003). In addition, a theoretical justification of the possibility was made of acquainting students with the elements of non-Euclidean geometries and methodological support of non-formal education. Practical measures in this direction have led to the creation of elective programs in non-Euclidean geometries (Gorshkova & Titova, 2005; Titova, 2006). When acquainting pupils with the elements of non-Euclidean geometries, Lénárt's approach can be useful, proposing to study the properties of plane and spherical geometry by means of comparative geometry, through direct experiments with practical tools (Lénárt, 2005; Lénárt, 2021).

## CONCLUSIONS AND PROSPECTS OF FURTHER RESEARCH

The above material indicates the possibility of mastering the basic concepts of non-Euclidean geometries by students studying in classes with in-depth study of mathematics. At the basic level, these concepts can be learned intuitively using their description, characteristics of the main properties and giving relevant examples. At the profile level, you can enter the distance axioms between the points of the metric space, examples of metric spaces and the corresponding numerical characteristics of basic geometric concepts.

In the further research it is necessary to continue in the direction of studying properties of flat arrangement of points, using thus definition of the angle formed by three points of metric space.

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#### ФОРМУВАННЯ ПОНЯТТЯ КУТА ЗАСОБАМИ МЕТРИЧНОЇ ГЕОМЕТРІЇ НА ГЕОМЕТРИЧНОМУ МАТЕРІАЛІ 9 КЛАСУ

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**Анотація.** Робота полягає у вивченні можливості застосування засобів метричної геометрії для формування основних геометричних понять при вивченні геометрії у закладах середньої освіти. Використання метричної геометрії відкриває шлях до знайомства учнів з елементами неевклідових геометрій як на інтуїтивному, так і на аксіоматичному рівнях. У роботі, на основі означення числової характеристики кута утвореного трьома точками метричного простору, дано альтернативне означення прямолінійного розміщення точок. За допомогою числової характеристики легко отримуються означення прямого та розгорнутого кутів. Наведений у роботі матеріал можна використовувати на уроках геометрії починаючи з 9-го класу, та у позакласній роботі з учнями які навчаються у класах з поглибленим вивченням математики.

**Формулювання проблеми.** Матеріал даної роботи стосується, у основному, викладання математики у класах з поглибленим вивченням математики. Сучасний стан математичної освіти ставить питання про необхідність ознайомлення учнів з основними поняттями та фактами неевклідових геометрій. Зробити це безпосередньо звертаючись до фактичного матеріалу таких геометрій досить складно, зважаючи на значний рівень його формалізації. У даній роботі, для вирішення цього питання автори пропонують використати засоби метричної геометрії, як найбільш наближеної до шкільного курсу геометрії. Пропонується розпочати цю роботу з формування узагальнених основних геометричних понять та об'єктів, таких як точка, кут, прямолінійне розміщення точок.

**Матеріали і методи.** Результати роботи отримані на підставі аналізу діючих підручників з математики для класів з поглибленим її вивченням, підручників з геометрії та математичного аналізу закладів вищої освіти, наукових публікацій та апробовані при читанні відповідного спецкурсу студентам спеціальності «014.04 Середня освіта (Математика)» магістерського рівня вищої освіти.

**Результати.** На основі запропонованого означення кута як упорядкованої трійки точок отримані аналоги класичних геометричних співвідношень. Ці аналоги допускають демонстрацію елементів неевклідових геометрій засобами елементарної геометрії.

**Висновки.** Аналітичний апарат метричної геометрії дає можливість сформувати узагальнене розуміння основних геометричних понять, таких як точка, кут, відстань між точками, прямолінійне розміщення точок.

**Ключові слова:** точка, кут, відстань, метрика, метричний простір.

