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ВКЛЮЧЕННЯ ТЕМИ «НАЙПРОСТІШІ ФУНКЦІОНАЛЬНІ РІВНЯННЯ» В МОДЕЛЬНІ ПРОГРАМИ ВИВЧЕННЯ ПРЕДМЕТУ «АЛГЕБРА І ПОЧАТКИ АНАЛІЗУ»

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АНОТАЦІЯ

Постановка проблеми. *Дослідження питання включення теми «Найпростіші функціональні рівняння» в модельні навчальні програми вивчення предмету «Алгебра і початки аналізу» для профільних класів з поглибленим вивченням математики. Модельна навчальна програма вивчає орієнтовну послідовність досягнення очікуваних результатів навчання, зміст предмета або інтегрованого курсу та види навчальної діяльності здобувачів освіти. Включання вказаної теми має на меті розпочати творче осмислення функціональних зв'язків, існуючих в реальних системах і процесах, зокрема, екологічних, економічних та соціальних.*

Матеріали та методи. *Теоретичний метод аналізу методичної та навчальної літератури з досліджуваного питання; порівняльний аналіз для усвідомлення різних поглядів на проблему; систематизація та узагальнення для створення рекомендацій змісту запропонованої теми, а також формулювання висновків та інтегрування педагогічного досвіду авторів, які викладають відповідні дисципліни в закладах освіти різних рівнів.*

INCLUSION OF THE TOPIC «THE SIMPLEST FUNCTIONAL EQUATIONS» IN THE MODEL PROGRAMS FOR STUDYING THE SUBJECT «ALGEBRA AND THE BEGINNINGS OF ANALYSIS»

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ABSTRACT

Formulation of the problem. *Analysis of the issue of including the topic "The simplest functional equations" in the model curricula for studying the subject "Algebra and the beginnings of analysis" for specialized classes with in-depth study of mathematics. A model curriculum studies the approximate sequence of achieving the expected learning outcomes, the content of the subject or integrated course, and the types of students' educational activities. The inclusion of this topic aims to start a creative understanding of functional relationships existing in real systems and processes, in particular, ecological, economic, and social ones.*

Materials and methods. *Theoretical method of analysis of methodical and educational literature on the researched issue; comparative analysis to understand different views on the problem; systematization and generalization to create recommendations for the content of the proposed topic, as well as formulating conclusions and integrating the pedagogical experience of authors who teach relevant disciplines in educational institutions of various levels.*

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Результати. *Запропоновано можливий зміст теми «Найпростіші функціональні рівняння» в модельні програми вивчення предмету «Алгебра і початки аналізу», приклади для пояснення викладачем і закріплення учнями. Для деяких прикладів запропоновані різні підходи їх розв'язання; надано зручні таблиці для пошуку учнями частинних розв'язків деяких видів функціональних рівнянь.*

Висновки. *Автори вважають, що тема «Найпростіші функціональні рівняння» буде корисною і сприйнятною для вивчення в межах предмету «Алгебра і початки аналізу» учнями профільних класів з поглибленим вивченням математики. В межах одинадцятирічної шкільної освіти, зрозуміло, часу на вивчення цієї теми знайти було неможливо за причини насиченості і щільності необхідного для вивчення матеріалу. Але в дванадцятирічній Новій Українській Школі, зазначеною більш глибокою диференціацією профільного навчання, тема «Найпростіші функціональні рівняння» може зміцнити фундаментальність математичної освіти в класах з поглибленим вивченням математики, інформатики тощо. Подальші дослідження в даному напряму можуть стосуватися методики розв'язання найпростіших рекурентних рівнянь.*

КЛЮЧОВІ СЛОВА: функція однієї змінної; функціональні рівняння; загальний розв'язок функціонального рівняння; частинний розв'язок функціонального рівняння; ін'єктивність функції.

Results. *The possible content of the topic "The simplest functional equations" in the model programs for studying the subject "Algebra and the beginnings of analysis", examples for an explanation by the teacher, and confirmation by students are proposed. For some examples, different approaches to their solution are proposed; convenient tables are provided for students to find partial solutions to some types of functional equations.*

Conclusions. *The authors believe that the topic "The simplest functional equations" will be useful and acceptable for studying within the scope of the subject "Algebra and the beginnings of analysis" by students of specialized classes with an in-depth study of mathematics. Within the eleven-year school education, of course, it was impossible to find time to study this topic due to the saturation and density of the material necessary for study. But in the twelve-year New Ukrainian School, marked by a deeper differentiation of specialized education, the topic "The simplest functional equations" can strengthen the fundamentality of mathematical education in classes with in-depth study of mathematics, computer science, etc. Further research in this direction may concern the method of solving the simplest recurrent equations.*

KEYWORDS: function of one variable; functional equations; general solution of a functional equation; partial solution of a functional equation; injectivity of a function.

INTRODUCTION

Formulation of the problem. Analysis of current research. The goal of the research group was to study the issue of including the topic "The simplest functional equations" in the model programs for studying the subject "Algebra and the beginnings of analysis" for specialized classes with an in-depth study of mathematics. This subject can be viewed from the point of view of its integrity, or can be viewed as an integrated one consisting of the subjects "Algebra" and "Introduction to Analysis". Both points of view emphasize relevant content, internal and intersubjective connections.

On the one hand, the topic "The simplest functional equations" is an intersection of the meaningful lines "equation", "function", and, on the other hand, it can be considered in the future as a propaedeutic topic for studying the theory of ordinary differential and difference equations in higher education. Here, in particular, students can be explained what differential and difference equations are, and generally speaking, that they are representatives of functional equations. In addition, functional equations can be considered as a foundation for building models in ecology (Matvieieva et al., 2019), mathematical economics (Zhebka et al., 2006), sociology and other sciences. Today, functional equations in school education, as a rule, are studied at the optional level (when preparing for mathematical Olympiads of various levels) (Voronyi, 2010; Pihtar, 2008; Pihtar, 2015; Serdiuk, 2019), as well as in the research of members of the Small Academy of Sciences (Zazvirska, 2013), that is, this topic is not studied by all students in a mandatory manner, but is studied selectively by those interested in it. According to the authors, this impoverishes school mathematics education, in particular, its professional orientation.

On the other hand, the discipline "Functional equations" appeared in a number of universities among the courses for students specialized in Mathematics. The study guide (Fedak, 2018), written in accordance with the curriculum of this discipline, contains the basic methods of solving functional equations and linear difference equations, but the level of presentation of theoretical aspects and the method of presenting the material are quite difficult for schoolchildren, even high school students, to understand. Issues of methods of solving certain types of functional equations, in particular, Cauchy, D'Alembert, Lobachevsky equations and the simplest recurrent equations are studied by V.F. Davydovych (Davydovych, 2018). Many scientific researches today are devoted to the stability of functional equations (Rassias et al., 2012; Moslehian et al., 2007; Moszner, 2009; Saadati et al., 2011; Brzdek et al., 2011; Noori et al., 2021) and their applications (Russias, 2003; El-Hady,2019).

METHODS OF RESEARCH

The basis for achieving the goal was the theoretical method of analyzing methodical and educational literature on the researched issue. Comparative analysis was also used to understand different views on the problem. Systematization and generalization became tools for creating recommendations for the content of the proposed topic, as well as formulating conclusions and integrating the pedagogical experience of authors who teach relevant disciplines in educational institutions of various levels.

RESULTS OF RESEARCH

The study of the simplest functional equations can be carried out in different trajectories. And what is proposed in this article is only one possible approach. The specified topic can become a link between algebra and the beginnings of analysis, based on a second point of view on the specified educational subject. The specified topic may include the following questions for study:

I. Elementary definition of a functional equation. A function that satisfies the corresponding functional equation.

When studying this issueis is possible to consider the following examples.

Exsmple 1. A function $u(x)$ is defined for $\forall x \in \mathbb{N}$ and satisfies conditions:

$$\Delta u = u(x) - \varphi(x-1) = \cos(x-1)\alpha,\ u(1) = 9. \tag{1.1}$$

Find $u(x)$.

Let's consider possible solving. Equalities follow from the condition of the example:

$$u(2) = u(1) + \cos\alpha\ ,$$

$$u(3) = u(2) + \cos 2\alpha\ ,$$

…………………

$$u(x)=u(x-1)+\cos(x-1)\alpha.$$

After adding their corresponding parts and summing similar terms, the following ratio can be obtained:

$$u(x)=u(1)+\cos\alpha+\cos 2\alpha+\ldots+\cos(x-1)\alpha.$$

Using a well-known formula $\cos\alpha+\cos 2\alpha+\ldots+\cos n\alpha=\dfrac{\cos\frac{\alpha}{2}-\cos\frac{(2n+1)\alpha}{2}}{2\sin\frac{\alpha}{2}}$ and the initial condition $u(1)=9$, it is possible to get the final answer: $u(x)=9+\dfrac{\cos\frac{\alpha}{2}-\cos\frac{(2x-1)\alpha}{2}}{2\sin\frac{\alpha}{2}}$.

The following task can be offered for selftraining:

A function $\varphi(x)$ is defines for $\forall x\in\mathbb{N}$ and satisfies the conditions: $\varphi(x+1)=\varphi(x)+2^x$, $\varphi(1)=3$. Find $\varphi(x)$. The answer: $\varphi(x)=2^{x+1}$.

Example 2 (of a research nature). Does a linear function exist $y=\varphi(x)$ that satisfies the equation for an arbitrary real x:

$$3\varphi(x+7)+\varphi(10-x)=11x+15. \tag{2.1}$$

1) Replace x by $x-7$ in (2.1):

$$3\varphi((x-7)+7)+\varphi(10-(x-7))=11(x-7)+15,$$
$$3\varphi(x)+\varphi(17-x)=11x-62. \tag{2.2}$$

2) Replace x by $17-x$ in (2.2):

$$3\varphi(17-x)+\varphi(x)=11(17-x)-62,$$
$$\varphi(x)+3\varphi(17-x)=-11x+125. \tag{2.3}$$

3) Multiply (2.2) by –3 and add to (2.3):

$$-9\varphi(x)-3\varphi(17-x)=-33x+186, \quad -8\varphi(x)=-44x+311,$$
$$\varphi(x)=\frac{44x}{8}-\frac{311}{8}, \quad \varphi(x)=\frac{11x}{2}-\frac{311}{8}.$$

The answer: there exists a linear function $\varphi(x)=\dfrac{11x}{2}-\dfrac{311}{8}$, that satisfies equation (2.1) for any $x\in\mathbb{R}$.

The following equation can be offered for selftraining:

Find a linear function $u(x)$, that satisfies the equation $2u(x+2)-5=2x-u(4-x)$. The answer: $u(x)=2x-\dfrac{11}{3}$.

Example 3 (of computational character). A function u satisfies the functional equation:

$$u(x+y)=u(x)+u(y) \text{ for all } \{x,\ y\}\subset\mathbb{Q}, \tag{3.1}$$

$u(9)=-e^2$. Find $u\left(\dfrac{5}{6}\right)$.

1) It is obvious that the function $u(x)=kx$ is a solution of this equation, in particular for $x\in\mathbb{Q}$. In fact:

$$u(x+y)=k(x+y)=kx+ky=u(x)+u(y).$$

Let $y=x$. Then

$$u(2x)=2u(x). \tag{3.2}$$

2) The hypothesis is: $u(nx)=nu(x)$, $\forall x\in\mathbb{Q}$, $\forall n\in\mathbb{N}$. For $n=1$ the identity is obvious.

$$u((n+1)x)=u(nx+x)=u(nx)+u(x)=nu(x)+u(x)=(n+1)u(x).$$

Then, according to the principle of mathematical induction, the hypothesis is correct.

3) Let's consider the relation

$$u(qx)=qu(x), \quad \forall\{x,q\}\subset\mathbb{Q}, \tag{3.3}$$

and prove its correctness for (3.1).

a) Let $y=0$ in (3.1), then $u(x)=u(x)+u(0)$. And $u(0)=0$, i.e. (3.3) is correct for $q=0$.

b) Let in (3.2) $x=\frac{z}{n}$, $n\in\mathbb{N}$, $z\in\mathbb{Q}$.Then

$$u(nx)=u(z)=nu\left(\frac{z}{n}\right)\Rightarrow u\left(\frac{1}{n}\cdot z\right)=\frac{1}{n}u(z). \tag{3.4}$$

It means that (3.3) is correct for $q=\dfrac{1}{n}$.

c) Let in (3.4) $x=\dfrac{z}{m}$, $m\neq n$, $z>0$, $m>0$. Then $z=mx$ and $u\left(\dfrac{m}{n}z\right)=\dfrac{1}{n}u(mx)=\dfrac{m}{n}u(x)$, i.e. (3.3) is true for all positive rational q.

d) Let $y=-x$ in (3.1). Then, taking into account a), we receive: $u(0)=u(x)+u(-x)=0$. Or $u(-x)=-u(x)$, and it means that the function $u(x)$ is odd for all $q\geq 0$. Thus, $u(-qx)=-u(qx)=-qu(x)$, i.e. (3.3) is true for all rational x, q.
For x = 1 in (3.3) we have:

$$u(q)=qu(1), \tag{3.5}$$

here $u(1)$ is a number.

e) If in (3.5) $x=q,\quad k=u(1)$, then $u(x)=kx$, and it means that the function $u(x)=kx$ is the only solution of the equation (3.1).

4) $u(9)=-e^2$.

$k\cdot 9=-e^2\Rightarrow k=-\left(\frac{e}{3}\right)^2$. Then $u(x)=-\left(\frac{e}{3}\right)^2 x$ is a particular solution of (3.1) for given initial conditions.

For solving certain classes of elementary functional equations, it is useful to know that one of the simplest functional equations is the Cauchy equation, which is defined for the function f: $\mathbb{R}\to\mathbb{R}$ as: $f(x+y)=f(x)+f(y)$. A function f that satisfies this equation is called additive. It is also possible to draw students' attention to the fact that the method of solving example №3 is often called the Cauchy method, for which particular solutions are known for $x\in\mathbb{Q}$ and $f(x)\in\mathbb{Q}$.

After this example, the following equation can be suggested for selftraining: $\psi(y)=\psi(x+y)-\psi(x),\quad \psi(10)=-\pi$. Find $\psi\left(-\frac{2}{7}\right)$. The answer: $\psi\left(-\frac{2}{7}\right)=\frac{\pi}{35}$.

II. Elementary functional equations, their particular solutions and features.

When studying this question, we can consider the following initial table 1 (Rassias et al., 2017). Note, that for the first dealing with elementary functional equations, if domain of definitions of variable x, variable y and $f(x)$, $f(y)$ is not indicated, then we assume that $x\in\mathbb{Q}$ and $f(x)\in\mathbb{Q}$, $y\in\mathbb{Q}$ and $f(y)\in\mathbb{Q}$.

Table 1.

Equation	Particular solutions	Notes
$f(x+y)=f(x)+f(y)$	$f(x)=kx$	A single family
$f(x\cdot y)=f(x)+f(y)$	$f(x)=c\ln\lvert x\rvert$, $f(x)=c\log_a\lvert x\rvert$, $f(x)=a\ln\lvert x\rvert+b$	Logarithmic equation. Some of the families
$f(x\cdot y)=f(x)\cdot f(y)$	$f(x)=\lvert x\rvert^{\alpha}$	Power equation. One of the families
$f(x+y)=f(x)\cdot f(y)$	$f(x)=e^{cx}$, $f(x)=a^x$, A degenerate solution $f(x)=0$	Exponential equation. One of the families

The continuation of this table will be considered below.

Example 4 (inductive solution). A numerical function satisfies the equality

$$u(x+y)=u(x)+u(y)+12xy,\quad \forall\{x,y\}\subset\mathbb{R}. \tag{4.1}$$

Find $u\left(\frac{7}{8}\right)$, if $u\left(\frac{1}{2}\right)=3$.

1) In the process of solving the simplest functional equations, induction plays an important role. It is obvious that the particular solution of this functional equation is a function $u_0(x)=6x^2$. In fuct:

$u_0(x+y)=6(x+y)^2=6x^2+12xy+6y^2=u_0(x)+u_0(y)+12xy$, which is consistent with (4.1).

2) Let

$$v(x)=u(x)-u_0(x). \tag{4.2}$$

Then $v(y)=u(y)-u_0(y)$,

$$v(x+y)=u(x+y)-u_0(x+y)=u(x)+u(y)+12xy-6(x+y)^2=$$
$$=u(x)+u(y)+12xy-6x^2-12xy-6y^2=u(x)-u_0(x)+u(y)-u_0(y)=$$
$$v(x)+v(y)$$, $\forall\{x,y\}\subset\mathbb{R}$, and, in particulas, $\forall\{x,y\}\subset\mathbb{Q}$.

3) From Example 2 we have $v(x)=kx$ ($\forall x\in\mathbb{Q}$). Then from (4.2): $u(x)=v(x)+u_0(x)=kx+6x^2$, $\forall x\in\mathbb{Q}$.

4) $u\left(\frac{1}{2}\right)=k\cdot\frac{1}{2}+6\cdot\left(\frac{1}{2}\right)^2=3$, $\frac{k}{2}+\frac{3}{2}=3\Rightarrow k=3$.

That is, $u(x)=3x+6x^2$ is a particular solution of this functional equation under given initial conditions.

5) $u\left(\frac{7}{8}\right)=3\cdot\frac{7}{8}+6\cdot\left(\frac{7}{8}\right)^2=\frac{231}{32}.$

After this type of example, you can draw students' attention to the fact that the structure of the general solution of functional equations is similar to the structure of the general solution of ordinary differential equations (which will be studied in higher education and used to build models of dynamic systems, processes, phenomena).

In order to understand a possible method of solving the fourth example, students can be offered to solve the following problem: $80xy=\varphi(x+y)-\varphi(x)-\varphi(y), \quad \varphi(0{,}25)=2$. Find $\varphi(0{,}8)$. The answer: $\varphi(0{,}8)=24$.

III. Expansion of the set of the simplest functional equations.

In this section one can consider table 2 (Rassias et al., 2017) which "absorbs" table 1. Note also that if domains of definition of x, y and $f(x)$, $f(y)$ are not indicated, then in the process of studying this material it is considered that $x\in\mathbb{Q}$ and $f(x)\in\mathbb{Q}$, $y\in\mathbb{Q}$ and $f(y)\in\mathbb{Q}$.

Table 2.

Functional equation	Elementary particular solution
$f(x+y)+f(x-y)=2f(x)$	$f(x)=kx+b$
$f(x+y)+f(x-y)=2f(x)f(y)$	$f(x)=\cos(kx)$
$f(x+y)+f(x-y)=2(f(x)+f(y))$	$f(x)=kx^2$
$f(x+y)+f(x-y)=2f(y)$	$f(x)=kx$
$f(x+y)\cdot f(x-y)=(f(x))^2$	$f(x)=c$
$f(x+y)-f(x-y)=4\sqrt{f(x)f(y)}$	$f(x)=kx^2$
$f(x+y)\cdot f(x-y)=(f(x))^2-(f(y))^2$	$f(x)=kx, \quad f(x)=c\sin(kx)$
$f(xy)=f(x)\cdot f(y), \quad \{x,y\}\subset$□	$f(x)=x^n$
$f(xy)=xf(y)+yf(y), \quad \{x,y\}\subset$□$^+$	$f(x)=x\ln x$
$f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}, \quad y\neq 0, \quad f(y)\neq 0$	$f(x)=x^n$
$f(x+y)=f(x)+f(y)$	$f(x)=f(1)\cdot x$ (one can draw students' attention to a different form of the particular solution compared to table 1)
$f(x+y)=f(x)\cdot f(y)$	$f(x)=(f(1))^x$ (one can draw students' attention to a different form of the particular solution compared to table 1)
$f(x)\cdot f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$	$f(x)=1\pm x^n$

In the process of solving elementary functional equations, it is useful to pay attention to the special properties of the functions with respect to which they are solved. The following example can be an illustration of this remark.

Example 5. A numerical function $u(x)$ satisfies the equality:

$$kx+u(x)=u(u(x)), \quad \forall x\in\mathbb{R}, \quad k\neq 0. \tag{5.1}$$

Solve the equation

$$u(u(x))=0. \tag{5.2}$$

1) $kx=u(u(x))-u(x), \quad x=\frac{u(u(x))-u(x)}{k}.$

If $u(x)=u(y)$, then $x=\frac{u(u(x))-u(x)}{k}=\frac{u(u(y))-u(y)}{k}=y$. It means that equality $u(x)=u(y)$ follows $x=y$, which indicates injectivity of $u(x)$ in this functional equation.

2) Injectivity of $u(x)$ follows that if $u(u(x))=u(u(y))$, then $u(x)=u(y)$, and $x=y$, i.e. (5.2) can have at most one root.

3) The obvious root of (5.1) is $x=0$. Really:

$$k\cdot 0+u(0)=u(u(0)), \quad u(0)=u(u(0)), \quad u(0)=0\text{, then } u(u(x))=0.$$

It follows from point 2) that this is the only root of the equation (5.2).

Example 6. A function $u(y)$ satisfies the equation:

$$u(y+2)=u(y+1)+2y+3, \quad \forall y\in\mathbb{R}. \tag{6.1}$$

Find $u(2864)$, if $u(0)=0$.

The first method. With this formulation of the problem, it is possible to consider the proposed functional equation only for natural values of the argument. Consider the following sequence:

$$p_n = u(n) - u(n-1), \quad n \in \mathbb{N}.$$

Then

$$\sum_{k=n-1}^{1}\left(u(k+1)-u(k)\right)=\left(u(n)-u(n-1)\right)+\left(u(n-1)-u(n-2)\right)+\ldots+\left(u(1)-u(0)\right)=$$

$$=u(n)=p_n+p_{n-1}+\ldots+p_1=\sum_{k=n}^{1}p_k .$$

For $\{p_n\}$ this equation has the form

$$u(n+2)=u(n+1)+2n+3.$$

Then

$$u(n+2)-u(n+1)=2n+3,$$

$$p_{n+2}=2n+3,$$

$$u(n+1)-u(n)=2(n-1)+3=2n+1,$$

$$p_{n+1}=2n+1,$$

$$p_{n+2}-p_{n+1}=(2n+3)-(2n+1)=2.$$

It means that $\{p_n\}$ is an arithmetic progression with $p_1=1$ and difference $d = 2$. Then $u(n)=S_n=\frac{2p_1+(n-1)\cdot 2}{2}\cdot n=n^2$ and $u(2864)=2864^2=8202496$.

The second method. It is easy to see that the particular solution of equation (6.1) is the function $u_0(y)=y^2$. In fact:

$$u_0(y+1)=(y+1)^2=y^2+2y+1,$$

$$u_0(y+2)=(y+1)^2+2y+3=y^2+2y+1+2y+3=(y+2)^2.$$

Let $y+1=t$. Then

$$u(t+1)=u(t)+2t+1.$$

Consider a function $v(t)$ that $u(t)=v(t)+u_0(t)$. Then:

$$v(t)=u(t)-u_0(t),$$

$$v(t+1)=u(t+1)-u_0(t+1)=u(t)+2t+1-(t+1)^2=$$

$$= u(t) + 2t + 1 - t^2 - 2t - 1 = u(t) - u_0(t) = v(t), \quad \forall t \in \mathbb{R}.$$

These considerations lead to the conclusion that the function $v(t)$ is periodic with the main period $T=1$ along the whole number line. Therefore, the solutions of this functional equation are all periodic functions with period $T=1$, defined on the whole number line, and only them.

Then the general solution of this functional equation has the form:

$$u(t)=v(t)+y^2, \text{ тобто } u(y+1)=v(y+1)+y^2,$$

where $v(y+1)$ is an arbitrary periodic function defined on the whole number line with period $T=1$. Initial condition $u(0)=0$ means that $v(0)=u(0)-u_0(0)=0$. Then for all integer y we have

$$u(y)=y^2+v(0)=y^2.$$

Thus, $u(2864)=2864^2=8202496$.

IV. The simplest functional equations solutions of which are sequences (particular cases of functions).

Example 7. A sequence $\{b_k\}$ is defined as:

$$b_1 = 4, \quad b_{k+1} = \begin{cases} 4b_k, \ if \ k = 2n, \ n \in \mathbb{N}, \\ b_k + 4, \ if \ k = 2n-1, \end{cases}$$

i.e. $b_2=4+4=8; b_3=8\cdot 4=32; b_4=32+4=36; b_5=4\cdot 36=144; b_6=144+4=147\ldots$ Find b_{22715}.

1) Consider the subsequence $b_1, b_3, b_5, \ldots$ Let $c_1=b_1, c_2=b_3, c_3=b_5, \ldots, c_m=b_{2m-1}$. Then $b_{22715}=c_{11358}$.
2) Functional equation for odd sequence numbers $\{b_k\}$ has the form:

$$b_{2k+1}=4b_{2k}=b_{(2k-1)+1}=4(b_{2k-1}+4)=4b_{2k-1}+16.$$

Then, respectively, functional equation for $\{c_m\}$ will be:

$$c_{m+1}=4c_m+16. \tag{7.1}$$

3) Let $c_m=d_m-\frac{16}{3}$, where $\{d_m\}$ is a new auxiliary sequence. Then $d_m=c_m+\frac{16}{3}$, a $c_{m+1}=d_{m+1}-\frac{16}{3}$.

And equation (7.1) is transformed to:

$$d_{m+1}-\frac{16}{3}=4\left(d_m-\frac{16}{3}\right)+16,$$

$$d_{m+1}=4d_m-\frac{4\cdot16}{3}+\frac{3\cdot16}{3}+\frac{16}{3}, \text{ тобто } d_{m+1}=4d_m,$$

It means that the sequence $\{d_m\}$ is a geometrical progression with the first element $d_1=c_1+\frac{16}{3}$ and denominator $q=4$. Thus $d_m=d_1\cdot4^{m-1}$.

$$c_m+\frac{16}{3}=\left(c_1+\frac{16}{3}\right)\cdot4^{m-1},\ c_m=c_1\cdot4^{m-1}+\frac{16}{3}\left(4^{m-1}-1\right).$$

It is obvious that $\left(4^{m-1}-1\right)\vdots3$, $c_1=b_1=4$ (according to the condition of the problem), then

$$b_{22715}=c_{11358}=4\cdot4^{11358-1}+\frac{16}{3}\left(4^{11357}-1\right)=4^{11358}+\frac{16}{3}\left(4^{11357}-1\right).$$

Remark 1. The first method of solving example 6 can be motivation for studying section IV.

Remark 2. As exercises for understanding the ideas of solving examples 4, 5 and 6, examples similar to the considered ones can be offered.

CONCLUSIONS AND PROSPECTS OF FURTHER RESEARCH

The material of the article was partially reviewed by the authors in educational institutions, where the complexity, accessibility, perception level and usefulness of the issues were analyzed. The authors believe that the topic "The simplest functional equations" will be useful and acceptable for studying within the scope of the subject "Algebra and the beginnings of analysis" by students of specialized classes with an in-depth study of mathematics.

Within the eleven-year school education, of course, it is impossible to find time to study this topic due to the saturation and density of the material necessary for study. But in the twelve-year-old New Ukrainian School, marked by a deeper differentiation of specialized education, the topic "The simplest functional equations" can strengthen the fundamentality of mathematical education in classes with in-depth study of mathematics, computer science, etc.

REFERENCES (TRANSLATED AND TRANSLITERATED)

1. Matvieieva, I., Tikhonova, V., & Groza, V. (2019). Application of Box Models in Radioecology. *Interdisciplinary Studies of Complex Systems,* 14, 53-57. https://doi.org/10.31392/iscs.2019.14.053.
2. Zhebka, V., Tykhonova, V., Leshchynskii, O., & Groza, V. (2006). *Dyferentsialni rivniannia v ekonomitsi [Differntial Equation in Economy]*. *Manual*. Kyiv: Delta (In Ukrainian).
3. Voronyi, O. (2010). *Funktsionalni rivniannia v olimpiadnii matematytsi [Functional equations in Olympiad mathematics] Methodical guide*. Kirovograd (In Ukrainian).
4. Pihtar, M. P. (2008). Metodika robotu ta sustema zadach z temi "Funktsionalni rivnyannya" [Methodology of work and system of problems on the topic "Functional equations"]. *Mathematics in the school*, 7-8, 48-56. (In Ukrainian).
5. Pihtar, M. P. (2015). Deiaki osoblyvosti provedennia hurtkovyh zaniat z rozviazannia olimpiadnyh zadach z matematyky v pedahohichnyh universytetah [Some features of conducting group classes on solving Olympiad problems in mathematics in pedagogical universities]. *Scientific journal of the M.P. Drahomanov NPU. Series 3. Physics and mathematics in higher and secondary school.* (In Ukrainian).
6. Serdiuk, Z.O. (2019). Facultatyvnyi kurs "funktsionalni rivniannia" dlia uchniv starshoii profilnoii shkoly [Optional course "functional equations" for students of a senior specialized school]. *Proceedings of the Cherkasy National University named after Bohdan Khmelnytskyi. "Pedagogical sciences" series, 3. 103-107.* (In Ukrainian).
7. Zazvirska , O.A. (2013). *Funktsionalni rivniannia. [Functional Equations].* Kirovohrad Regional Branch of the Small Academy of Sciences of Ukraine. https://docs.google.com/document/d/1HDnQ0xFe78_pIa17AVgUAoDXSyTNcbexLp9CVhvCGd0/edit (In Ukrainian).
8. Fedak, I.V. (2018). *Funktsionalni rivniannia. [Functional Equations]. Manual.* Ivano-Frankivsk: PNU (In Ukrainian).
9. Davydovych, V.F. (2019). Functsionalni rivniannia: pryklady, zastosuvannia, metody rozviazuvannia [Functional equations: examoles, applications, methods of solving]. *Proceedings of Seventh International Scientific-Practical Conference "Mathematics in Modern Technical University".* Vinnytsia: Publisher FOP Kushnir Yu.V. (In Ukrainian).
10. Russias, T.M., & Brzdek, J. (2012). *Functional Equations in Mathematical Analysis.* New York, NY, USA: Springer.
11. Moslelehian, M.S., & Russias, T.M. (2007). Stability of Functional Equations in Non-Archimedean Spaces. *Applicable Analysis and Discrete Mathematics,* 1, 325-334.
12. Moszner, Z. (2009). On the stability of functional equations. *Aequantiones Mathematicae*, 77. 33-88.
13. Saadati, R., Zohdi, M.M., & Vaezpour, S.M. (2011). Nonlinear L-Random Stability of an ACQ Functional Equation. *Journal of Inequalities and Applications.* https://doi.org/10.1155/2011/194394.
14. Brzdek, J., Chudziak, J., & Pales, Z. (2011) A fixed point approach to stability of functional equations. *Nonlinear Analysis: Methods & Applications.* https://doi.org/10.1016/j.na.2011.06.052.
15. Noori, B., Moghimi, V.B., Najati, A., Park, C., & Lee, J.R. (2021). On superstability of exponential functional equations. *Journal of Inequalities and Applications.* https://doi.org/10.1186/s13660-021-02615-w.
16. Russias, T.M. (2003). *Functional Equations, Inequalities and Applications.* Dordrecht, The Netherlands^ KluwerAcademic Publishers.
17. El-Hady, E.-S. (2019). On Some Functional Equations with Applications in Networks. *In book: Frontiers in Functional Equations and Analytic Inequalities*. Springer, 295-308*.* https://doi.org/10.1007/978-3-030-28950-8_17.
18. Rassias, J.M., Thandapani, E., Ravi, K., & Senthil Kumar, B. V. (2017). *Functional Equations and Inequalities*. https://doi.org/10.1142/10224.