Scientific journal
PHYSICAL AND MATHEMATICAL EDUCATION
Has been issued since 2013.
Науковий журнал
ФІЗИКО-МАТЕМАТИЧНА ОСВІТА
Видається з 2013.

p-ISSN 2413-1571
e-ISSN 2413-158X
DOI: 10.31110/2413-1571
https://fmo-journal.org/

DOI **10.31110/2413-1571-2022-033-1-004** UDK **514.84**

FORMATION OF MODERN MATHEMATICAL APPROACH TO SOLVING PROBLEMS OF PHYSICS

ФОРМУВАННЯ СУЧАСНОГО МАТЕМАТИЧНОГО ПІДХОДУ ДО ВИРІШЕННЯ ПРОБЛЕМ ФІЗИКИ

Тетяна ОБІХОД ✉
Інститут ядерних досліджень НАН України, Київ, Україна
obikhod@kinr.kiev.ua
https://orcid.org/0000-0003-1103-4006

Tetiana OBIKHOD ✉
Institute for Nuclear Research NAS of Ukraine, Kyiv, Ukraine
obikhod@kinr.kiev.ua
https://orcid.org/0000-0003-1103-4006

АНОТАЦІЯ

Постановка проблеми. *Точні дослідження бозона Хіггса, суперсиметричних частинок, магнітного моменту мюона, електричного дипольного моменту електрона, аномалій аромату демонструють відхилення від Стандартної моделі. Вони пов'язані з новим розумінням квантової теорії поля через об'єднання гравітації з фізикою елементарних частинок в рамках теорії струн - потужного інструменту, який змінив картину теорії. Стаття присвячена вивченню нової фізики через ці дві складові. Спочатку ми розглянули фізику елементарних частинок з точки зору останніх експериментальних даних, а потім перейдемо до математичного апарату теорії струн.*

Матеріали та методи. *Теорія Янга-Мілса N = 2 є аналогом гетеротичної струни, що визначається у десятивимірному просторі: чотири звичайні просторово-часові координати та шість додаткових вимірів, відомих як многовиди Калабі-Яу у зваженому проективному просторі. Ми досліджували многовиди Калабі-Яу в термінах як диференціальних форм, так і рефлексивних багатогранників, щоб отримати інформацію про елементарні частинки. Для подальшої роботи з многовидами Калабі-Яу були використані диференціальні форми для обчислення груп когомологій та рефлексивні багатогранники для обчислення чисел Ходжа. Ми використали два визначення загальних властивостей торичних многовидів: як гіперповерхні в термінах диференціальних форм і як проективний простір у термінах рефлексивного багатогранника. Потім ми досліджували гратчасті многогранники Δ, які породжують сімейства гіперповерхонь Калабі-Яу у зваженому проективному просторі РΔ. Такі багатогранники допускають комбінаторну характеристику і називаються рефлексивними.*

Результати. *Порівняння двох підходів до опису многовиду Калабі-Яу як комплексного многовиду і як зваженого проективного простору привело нас до висновку про еквівалентність цих двох трактувань у контексті обчислення характеристики Ейлера. Оскільки характеристикою Ейлера для фізики елементарних частинок є кількість поколінь кварків і лептонів, вибір многовидів Калабі-Яу з відповідними топологічними властивостями є однією з актуальних проблем сучасної фізики. Необхідно підкреслити, що важливим результатом нашої роботи є збіг значення ейлерової характеристики, знайденої в термінах когомології Дольбо і в термінах рефлексивного поліедра. Отримана інформація про топологічні інваріанти необхідна для прогнозування кількості поколінь у фізиці елементарних частинок.*

Висновки. *Хоча єдиної теорії всіх взаємодій поки що не знайдено, однак певні аспекти, пов'язані з трактуванням єдиної теорії всіх взаємодій з точки зору сучасної математики, дають свої вагомі результати. Тому використання та розробка апарату алгебраїчної геометрії для пошуку топологічних інваріантів, що мають значення спостережуваних у фізиці, є актуальним завданням.*

КЛЮЧОВІ СЛОВА: Стандартна модель; многовид Калабі-Яу; торичні многовиди; диференційна форма; зважений проективний простір; рефлексивний багатогранник; когомологія Дольбо; Характеристика Ейлера.

ABSTRACT

Formulation of the problem. *Precision studies of the Higgs boson, supersymmetric particles, the magnetic moment of the muon, electric dipole moment of the electron, flavor anomalies demonstrate the deviation beyond Standard Model. They are connected with a new understanding of quantum field theory through the unification of gravity with particle physics in the framework of string theory - the powerful instrument, which has changed the theory picture. The article is devoted to the study of new physics through these two components. First, we considered particle physics in terms of the latest experimental data and then moved on to the mathematical apparatus of string theory.*

Materials and methods. *The N = 2 Yang-Mills theory is the heterotic string analog determined in ten-dimensional space: four usual space-time coordinates and six extra dimensions, known as Calabi-Yau manifold in weighted projective space. We studied the Calabi-Yau manifold in terms of both differential forms and reflexive polyhedra to extract the elementary particle information. For further work with Calabi-Yau manifolds, differential forms for calculation of cohomology groups and reflexive polyhedra for calculation of Hodge numbers were used. We used two definitions of general properties of toric varieties: hypersurfaces in terms of differential forms and projective space in terms of reflexive polyhedral. Then we investigated lattice polyhedra Δ which gives rise to families of Calabi-Yau hypersurfaces in weighted projective space, PΔ. Such polyhedra admit a combinatorial characterization and are called reflexive polyhedra.*

Results. *The comparison of two approaches to the description of Calabi-Yau manifold as a complex manifold and as weighted projective space led us to the conclusion about the equivalence of these two treatments in the context of calculation of the Euler characteristic. As Euler's characteristic for elementary particle physics is the number of generations of quarks and leptons, the selection of Calabi-Yau manifolds with appropriate topological properties is one of the urgent problems of modern physics. It is necessary to stress that the important result of our paper is the coincidence of the value of the Euler characteristic, found in terms of Dolbeault cohomology and terms of reflexive polyhedral. The obtained information about topological invariants is necessary for predicting the number of generations in particle physics.*

Conclusions. *Although a unified theory of all interactions has not yet been found, however, certain aspects related to the interpretation of the unified theory of all interactions in terms of modern mathematics give their significant results. Therefore, the use and development of the apparatus of algebraic geometry for finding topological invariants that have the value of observables in physics is an urgent task.*

KEYWORDS: Standard Model; Calabi-Yau manifold; toric varieties; differential form; weighted projective space; reflexive polyhedral; Dolbeault cohomology; Euler characteristic.

INTRODUCTION

Formulation of the problem. The discovery of the Standard Model (SM) Higgs boson at electroweak scale was a triumph of field theory. But the existence of hierarchy problems, dark matter, and dark energy, the recent experimental data (LHCb Collaboration, 2021; Crivellin et al., 2021; Bobeth et al., 2021; Muon g-2 collab., 2021) has led us to the most important channels for

Для цитування:	Obikhod T. Formation of modern mathematical approach to solving problems of physics. *Фізико-математична освіта*, 2022. Том 33. № 1. С. 26-29. DOI: 10.31110/2413-1571-2022-033-1-004
	Obikhod, T. (2022). Formation of modern mathematical approach to solving problems of physics. *Фізико-математична освіта*, 33(1), 26-29. https://doi.org/10.31110/2413-1571-2022-033-1-004
For citation:	Obikhod, T. (2022). Formation of modern mathematical approach to solving problems of physics. *Physical and Mathematical Education*, 33(1), 26-29. https://doi.org/10.31110/2413-1571-2022-033-1-004
	Obikhod, T. (2022). Formation of modern mathematical approach to solving problems of physics. *Fizyko-matematychna osvita – Physical and Mathematical Education*, 33(1), 26-29. https://doi.org/10.31110/2413-1571-2022-033-1-004

✉ Corresponding author

the search for new physics: a)B-decay anomalies, connected with the transition between quarks, b→sll, strongly suppressed by SM showed 3.1σ from SM predictions; b) BaBaR, Belle and LHCb analysis of charged-current b→cτν decays showed the combined significance of roughly 3σ; c)combined measurements at Brookhaven and Fermilab of the magnetic moment of the muon presented the overall significance of 4.2 σ; d)Cabibbo-Kobayashi-Maskawa elements determined with unitarity relation demonstrated a deficit at the 3σ level. To resolve the problems of SM, we will consider particle physics in terms of the mathematical apparatus of superstring theory.

The N = 2 Yang-Mills theory is the heterotic string analog recently solved in the work of Seiberg and Witten (Seiberg & Witten, 1994). This theory is determined in ten-dimensional space: four usual space-time coordinates and six extra dimensions, known as Calabi-Yau manifold in weighted projective space. It turns out that physical information about matter content can be obtained from the study of geometric properties and the calculation of topological invariants of the space of extra dimensions that appears in string theory. So, we'll study the Calabi-Yau manifold in terms of differential forms and reflexive polyhedra to extract the information about the number of particle generation.

Analysis of current research. The emergence of string theory is associated with the analysis of experimental data on pion scattering, which was described in 1968 by J. Veneciano and M. Suzuki using beta functions. In 1970, Nambu J., Goto T., Nielsen H. and Sasskind L. put forward the idea of the interaction between pi-mesons through "an infinitely thin thread that oscillates", (Susskind, 1970). Thus, the theory of superstrings appeared, which describes elementary particles and the interactions between them. Scientists such as Green M., Schwartz J., Volkov DV, Kazakov DI (Green et al., 2012) associated with the development of this theory. In the 1990s, Witten E., Polchinsky J., and others discovered evidence for combining superstring theories into an 11-dimensional M-theory, (Susskind, 1970). Such theories are connected with Calabi-Yau space with properties necessary for preservation supersymmetry after compaction of ten-dimensional space on a six-dimensional torus.

The purpose of the article. The statement about the Yang-Mills theory presented in terms of heterotic string theory gives us the possibility to receive the problems of high energy theory in the mathematical language of string theory. So, the purpose of our paper is to present the equivalence of the calculation of topological invariants of algebraic geometry through the Dolbeault cohomology or in terms of reflexive polyhedral, which gives us the information of the number of generation in physics.

THEORETICAL BASES OF THE RESEARCH

In the future, we will work with a section of mathematics - topology. Topology is connected with bending, stretching, and compressing of the material without limit, but cannot create or destroy holes within shapes. So, a new concept called homology helps mathematicians connect topology with algebra. We will deal with the counting of a specific type of hole, which can be described as a closed and hollow space. The samples of one-dimensional holes of different types are presented in Fig. 1.

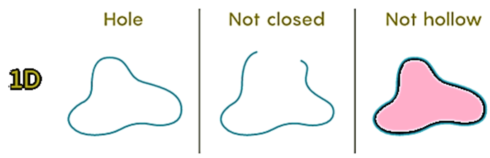

Fig. 1. One-dimensional holes of different types

To distinguish closed and hollow spaces it is necessary to know the information about its boundaries: the boundary of a one-dimensional line segment, solid triangle, and solid pyramid are the two points, the hollow triangle, and hollow pyramid correspondingly. Different types of holes are characterized by mathematicians by a chain complex. For example, a pyramid bounded by four triangles, six lines, and four points can be presented as a chain complex, Fig. 2.

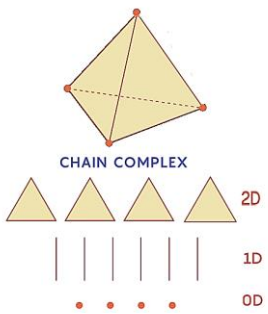

Fig. 2. Chain complex of the pyramid that gives the assembly instructions for a shape

So, any shape is built by grouping by dimension and arranging through gluing pieces of different dimensions. In mathematical language, we'll describe this procedure in the following way. Let's consider n-simplex σ, which is a continuous mapping of $\Delta_n \to X$ (X is the space), and consider the mapping $F_i : \Delta_{n-1} \to \Delta_n$ obtained by the restriction of linear embedding $R^n \to R^{n+1}$. The composition $\sigma \circ F_i$ is an (n−1)-simplex and is called the i-th face of σ. From these faces is constructed n-cochain with values in A (abelian group), which associates with each singular n-simplex the element from A. For such a cochain α we can define its coboundary: the (n + 1)-cochain dα,

$$d\alpha(\sigma) = \sum_{i=0}^{n+1} (-1)^i \alpha(\sigma \circ F_i) \cdot$$

So, the construction of any shape in X from the pyramid is presented through the complex

$$S(X)^{\bullet} : 0 \to S_X^0 \to \ldots \to S_X^n \to \ldots$$

called singular cochain complex of the space X. Its cohomology is called singular cohomology of X with values in A denoted by $H^i(X, A)$. The language of differential forms, simplexes for the description of extra-dimensional space – Calabi-Yau manifold will be used in the presented paper.

METHODS OF THE RESEARCH

For further work with Calabi-Yau manifolds, we will use two approaches: a) calculation of cohomology groups using differential forms b) calculation of Hodge numbers using reflexive polyhedra.

a) To describe complex manifolds of Calabi-Yau type, we'll consider differential forms on such manifolds with basic differentials

$$dz^j = dx^j + idy^j .$$

For (p,q)-forms

$$\omega = \omega_{i\ldots n\bar{i}\ldots\bar{n}} dz^i \wedge \ldots \wedge dz^n \wedge d\bar{z}^i \wedge \ldots \wedge d\bar{z}^n$$

the external derivatives have the form

$$\partial\omega = \frac{\partial}{\partial z^i}\omega_{i\ldots n\bar{i}\ldots\bar{n}} dz^i \wedge \ldots \wedge d\bar{z}^n, \quad \bar{\partial}\omega = \frac{\partial}{\partial \bar{z}^i} d\bar{z}^i \omega_{i\ldots n\bar{i}\ldots\bar{n}} dz^j \wedge \ldots \wedge d\bar{z}^n .$$

Dolbo's cohomology, (Griffiths & Harris, 2014), which characterize the type of manifold is defined by

$$H^{p,q} = \frac{\bar{\partial} - closd(p,q) - form}{\bar{\partial} - exact(p,q) - form} \bullet$$

b) Calabi-Yau manifold $X_d(\omega_1, \ldots, \omega_5)$ in weighted projective space

$$P^4_{\omega_1,\ldots,\omega_5} = P^4 / Z_{\omega 1} \times \ldots \times Z_{\omega 5},$$

with P^4 -four-dimensional projective space and Z_{ω_i} - the cyclic group is defined from the superpotential $W(\varphi_1, \ldots, \varphi_5) = 0$, (Candelas et al., 1990; Greene et al., 1991), which satisfies the homogeneity condition,

$$W(x^{\omega_1}\varphi_1, \ldots, x^{\omega_5}\varphi_5) = x^d W(\varphi_1, \ldots, \varphi_5), \quad d = \sum_{i=1}^{5}\omega_i, \ \varphi_{1,\ldots}\varphi_5 \in P^4_{\omega_1,\ldots,\omega_5} .$$

Such a variety is called toric one, Fig.3

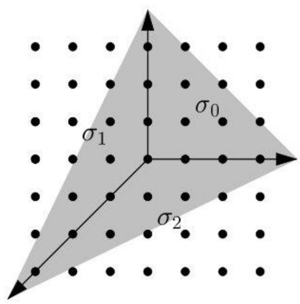

Fig. 3. Example of toric variety in P^2

which comes from the fans by gluing them all together. So, we can construct reflexive polyhedron, (Batyrev, 1993), from fans

$$\Delta(\bar{\varpi}) := \left\{ (x_1, \ldots, x_{n+1}) \in R^{n+1} \mid \sum_{i=1}^{n+1} \varpi_i x_i = 0, \ x_i \geq -1 \right\}.$$

Fans are combined from a set of convex cones σ in R^d. For $\vec{n}_1, \ldots, \vec{n}_r \in Z^d$, $\sigma = R_{\geq}\vec{n}_1 + \ldots + R_{\geq}\vec{n}_r$, *d* is equal to $d = \sum_{i=1}^{5}\omega_i$ - the sum of Calabi-Yau weights. For further consideration, we need the notion of a hypersurface - a manifold or an algebraic variety of dimension n − 1, which is embedded in an ambient space of dimension n. In general, it is a Euclidean space, an affine space, or a projective space. We are dealing with projective spaces. Most definitions assume the Calabi-Yau manifold is non-singular, but for singular Calabi–Yau's, the canonical bundle and canonical class may still be defined. V. Batyrev has proposed the following theorem, (Batyrev, 1993):

Theorem. Let Δ be n-dimensional integral polyhedron, P_Δ – the n-dimensional projective toric variety, F(Δ) - the family of hypersurfaces $\bar{Z}_f$ in P_Δ. Then the family F(Δ) of hypersurfaces in P_Δ consists of Calabi-Yau varieties with canonical singularities.

RESULTS OF RESEARCH

To distinguish between two shapes that are hard to visualize an invariant in the form of a single number exists, which doesn't vary after the change of an object's inessential features (if you stretch or distort some shape). Centuries ago mathematicians discovered, that a combination of the number of shapes plus the number of corners minus the number of edges always comes out the same. For example, a tetrahedron (with four triangles, four corners, and six edges), has the number 4 + 4 − 6 = 2. The same procedure can be done for Calabi-Yau hypersurface Z_f through the calculation of the Hodge-Deligne number $h^{n-2,1}(Z_f)$ of the cohomology group $H^{n-1}(Z_f)$ as follows

$$h^{n-2,1}(Z_f) = l(\Delta) - n - 1 - \sum_{\mathrm{codim}\,\Theta=1} l^*(\Theta).$$

where face $\Theta \subset \Delta$, $l(\Delta)$ – Mori generators, (Candelas et al. 1994) . The Euler characteristic for p,q=n is defined as the alternated sum of Hodge-Deligne numbers,

$$\sum_{i\geq 0}(-1)^i h^{p,q}(H_c^i(V)).$$

On the other hand, we can construct Euler's characteristics using Dolbo's cohomology. Knowing, that

$$\dim H_{\bar{\partial}}^{p,q}(M) = h^{p,q}$$

where $h^{p,q}$ - Hodge numbers, the Euler characteristic for a flat plane variety is determined from Hodge numbers as follows

$$\chi = \sum_{p,q=n}(-1)^{p+q} h^{p,q} = 2\left(h^{1,1} - h^{2,1}\right).$$

A number of generations $= \frac{1}{2}|\chi|$.

CONCLUSIONS AND PERSPECTIVES FOR A FURTHER RESEARCH

The comparison of two approaches to the description of Calabi-Yau manifold as a complex manifold and as toric variety led us to the conclusion about the equivalence of these two treatments in the context of calculation of the Euler characteristic. Topological invariants reflect the geometric properties of the manifold, and on the other hand, the Euler characteristic for elementary particle physics is the number of generations of quarks and leptons. Thus, Einstein's statement that "physics is geometry" is realized. Therefore, the selection of Calabi-Yau manifolds with appropriate topological properties is one of the urgent problems of modern physics. On the other hand, it is necessary to stress the important result of the coincidence of the value of the Euler characteristic, found in terms of Dolbeault cohomology or terms of reflexive polyhedral. Such a result leads us to the conclusion about the importance of modern mathematical approaches to physical problems.

REFERENCES (TRANSLATED AND TRANSLITERATED)

1. Batyrev, V. (1993). Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties. *Algebraic Geometry*, https://arxiv.org/abs/alg-geom/9310003.
2. Bobeth, C., Bordone, M., Gubernari, N., Jung, M., & Danny van Dyk. (2021). Lepton-flavour non-universality of B¯→D∗ℓν¯ angular distributions in and beyond the Standard Model. *Eur. Phys. J. C* 81, 984. https://doi.org/10.1140/epjc/s10052-021-09724-2.
3. Candelas, P., Lynker, M., & Schimmrigk R. (1990). Calabi-Yau Manifolds in Weighted P(4). *Nucl. Phys. B341*, 383-402. https://doi.org/10.1016/0550-3213(90)90185-G
4. Candelas, P., de la Ossa, X., & Katz, S. (1994). Mirror Symmetry for Calabi-Yau Hypersurfaces in Weighted P_4 and Extensions of Landau Ginzburg Theory. *Nucl.Phys. B450*. https://arxiv.org/abs/hep-th/9412117.
5. Crivellin, A., Manzari, C., Algueró, M., & Matias, J. (2021). Combined Explanation of the Z→b¯b Forward-Backward Asymmetry, the Cabibbo Angle Anomaly, and τ→μνν and b→sℓ+ℓ− Data. *Phys. Rev. Lett.,* 127, 011801. https://doi.org/10.1103/PhysRevLett.127.011801
6. Green, M., Schwarz, J., & Witten, E. (2012). Superstring theory. Vol. 1: *Introduction*. Cambridge University Press.
7. Green, M., Schwarz, J., & Witten, E. (2012). Superstring theory. Vol. 2: *Loop amplitudes, anomalies and phenomenology*. Cambridge University Press.
8. Greene, B.R., Roan, S.S., & Yau, S. T. (1991). Geometrie singularities and spectra of Landau-Ginzberg models. Commun. Math. Phys. 142, 245-259. https://doi.org/10.1007/BF02102062
9. Griffiths, P., & Harris, J. (2014). *Principles of Algebraic Geometry*. John Wiley & Sons.
10. LHCb Collaboration (2021). Tests of lepton universality using B0→K0Sℓ+ℓ− and B+→K∗+ℓ+ℓ− decays. LHCb-PAPER-2021-038. http://cds.cern.ch/record/2784954
11. Muon g-2 collab. (2021). Measurement of the Positive Muon Anomalous Magnetic Moment to 0.46 ppm. *Phys.Rev.Lett.,* 126, 141801. https://doi.org/10.1103/PhysRevLett.126.141801
12. Seiberg, N. & Witten, E. (1994). Electric-magnetic Duality, Monopole Condensation, and Confinement in N=2 Supersymmetric Yang-Mills Theory. *Nucl. Phys.,* B426, 19-52. https://doi.org/10.1016/0550-3213%2894%2990124-4
13. Susskind, L. (1970). Structure of Hadrons Implied by Duality. *Physical Review D*. 1 (4), 1182–1186. https://link.aps.org/doi/10.1103/PhysRevD.1.1182