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ЗАСТОСУВАННЯ ТЕОРІЇ М'ЯКИХ МНОЖИН ДО ОЦІНЮВАННЯ НАВИЧОК МАТЕМАТИЧНОГО МОДЕЛЮВАННЯ

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APPLICATION OF SOFT SETS TO ASSESSMENT OF MATHEMATICAL MODELLING SKILLS

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АНОТАЦІЯ

Формулювання проблеми. *Математичне моделювання є важливою складовою математичної освіти, оскільки застосування математичних теорій до практичних потреб повсякденного життя підвищує інтерес учнів до математики. Основні етапи процесу математичного моделювання включають аналіз заданої реальної проблеми, постановку задачі та побудову математичної моделі (математизацію), розв'язування і контроль моделі та впровадження математичних результатів у реальну ситуацію. Математизація характеризується найбільшою складністю серед етапів процесу ММ, оскільки вона передбачає глибокий процес абстрагування, якого не завжди легко досягти нефахівцю. Проте іноді перехід від розв'язування моделі до реального світу (контроль та/або реалізація моделі) також створює труднощі для студентів. Для ілюстрації цього зауваження наведено приклад.*

Матеріали і методи. *У цій роботі м'які множини використовуються як інструменти для розробки моделі параметричної оцінки людської діяльності та наведено приклад (оцінка працездатності футболістів) для ілюстрації її застосовності в реальних ситуаціях. М'яка множина, будучи параметризованою множиною підмножин універсальної множини дискурсу, є узагальненням концепції нечіткої множини, розробленої з метою параметричного поводження з існуючою невизначеністю. Теорія м'яких множин знайшла багато застосувань у кількох секторах людської діяльності, як-от прийняття рішень, скорочення параметрів, кластеризація даних, що стосуються неповноти тощо.*

Результати. *Модель оцінювання через м'які множини застосовується для оцінювання навичок математичного моделювання студентів за параметрами відмінно, дуже добре, добре, посередньо та неефективно. Вона служить як для оцінки загальної успішності класу, так і індивідуальної роботи кожного студента щодо виконання етапів процесу математичного моделювання.*

Висновки. *Побудована в статті модель дуже корисна в тих випадках, коли оцінка має якісні, а не кількісні характеристики, і її також можна застосувати до низки інших випадків для оцінки діяльності людини та/або машин (наприклад, комп'ютерних програм).*

КЛЮЧОВІ СЛОВА: *математичне моделювання; методи оцінювання; м'які множини; нечітка логіка.*

ABSTRACT

Formulation of the problem. *Formulation of the problem. Mathematical modelling is a very important component of mathematics education, because by applying the mathematical theories to practical needs of our everyday life increases the student interest for mathematics. The main steps of the mathematical modelling process include analysis of the given real world problem, formulation of the problem and construction of the mathematical model (mathematization), solution and control of the model and implementation of the final mathematical results to the real situation. Mathematization possesses the greatest difficulty among the steps of the MM process, because it involves a deep abstracting process, which is not always easy to be achieved by a non-expert. It is sometimes, however, the transition from the solution of the model to the real world (control and/or implementation of the model) that presents difficulties for students too. An example is given to illustrate this remark.*

Materials and methods. *In this paper soft sets are used as tools for developing a model for assessing human activities in a parametric manner and an example is presented (assessment of football players performance) to illustrate its applicability under real situations. A soft set, being a parametrized family of subsets of the universal set of the discourse, is a generalization of the concept of fuzzy set designed on the purpose of dealing with the existing uncertainty in a parametric manner. The theory of soft sets has found many and important applications to several sectors of the human activity like decision making, parameter reduction, data clustering and data dealing with incompleteness, etc.*

Results. *The soft set assessment model is applied for evaluating student mathematical modelling skills with respect to the parameters excellent, very good, good, mediocre, and failed. It serves both for assessing the general performance of a student class and the individual performance of each student with respect to the steps of the mathematical modelling process.*

Conclusions. *The constructed in this paper model is very useful in cases where the assessment has qualitative rather than quantitative characteristics and could also be applied to a variety of other cases for assessing human and/or machine (e.g. computer programs) activities.*

KEYWORDS: *mathematical modelling (MM); assessment methods; soft sets; fuzzy logic.*

INTRODUCTION

The process of modelling. The notion of a **system** has a very broad context. Roughly speaking, it can be defined as a set of interacting components forming an integrated whole. Examples of systems include the physical systems (the Earth, our solar system, the whole universe, etc.), social systems (our society, religions, countries and organizations, scientific communities, etc.), economic systems (companies, industries, etc.), biological systems like human or animal organizations, abstract systems (mathematical, philosophical, etc.), artificial systems designed by the humans (buildings, transportation means, etc.) and many others.

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The systems' **modelling** is a basic principle in engineering, in natural and in social sciences. When we face a problem concerning a system's operation (e.g. maximizing the productivity of an organization, minimizing the functional costs of a company, etc.) a **model** is required to describe and represent the system's multiple views. The model is a simplified representation of the basic characteristics of the real system including only its entities and features under concern.

There are several types of models in use according to the form of the corresponding problem ((Taha, 1967), section 1.3.1). The representation of a system's operation through the use of a **mathematical model** is achieved by a set of mathematical expressions (equalities, inequalities, etc.) and functions properly related to each other. The solutions provided by a mathematical model are more general and accurate than those provided by the other types of models. In cases, however, where a system's operation is too complicated to be described in mathematical terms (e.g. biological systems), or the corresponding mathematical relations are too difficult to deal with in providing the problem's solution, a **simulation model** can be used, which is usually constructed with the help of computers.

Mathematical Modelling in Education. Until the middle of 1970's mathematical modelling used to be mainly a tool in hands of scientists and engineers for solving the real world problems related to their disciplines (physics, industry, constructions, economics, etc.). The failure of the introduction of the "new mathematics" in school education, however, placed the attention of specialists on the use of the problem as a tool and motive to teach and understand better mathematics. The perceptions of this movement are mainly expressed through **Problem Solving**, where attention is given to the use of the proper heuristic strategies for solving mathematical problems (Voskoglou, 2012), and **Mathematical Modelling (MM) and Applications**, i.e. the solution of a particular type of problems .generated by real world situations.

One of the first who described the process of MM in such a way that it could be used for teaching mathematics was Pollak (Pollak, 1979). He represented the interaction between mathematics and the real world with the scheme shown in Figure 1, which is known as the **circle of modelling.**

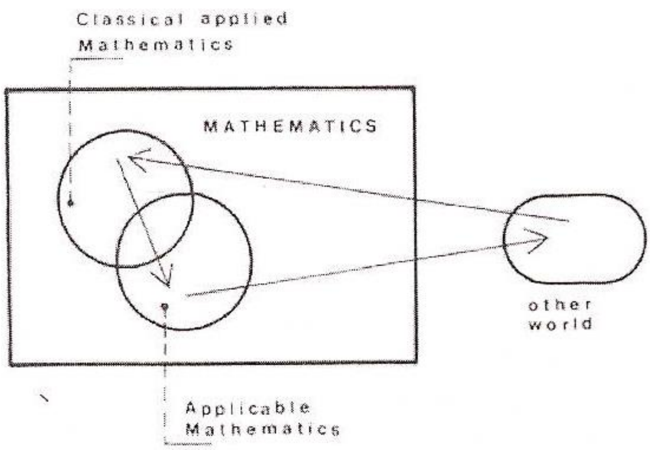

Fig. 1. The circle of modelling

According to the Pollak's scheme, in the "universe" of mathematics classical applied mathematics and applicable mathematics are two intersected but not equal to each other sets. In fact, there are topics from classical mathematics with great theoretical interest, but without any visible applications (although such applications it is possible to be found in future), while at the same time they are branches of mathematics with many practical applications, which are not characterized by many people as classical mathematics (e.g. statistics, fuzzy logic, fractals, linear programming etc.). The most important feature of Pollak's scheme, however, is the direction of the arrows, representing a looping between the real (other) world including all the other sciences and the human activities of everyday life and the "universe" of mathematics. Starting from a real problem of the real world we transfer to the other part of the scheme, where we use or develop suitable mathematics for its solution. Then we return to the real world interpreting and testing the mathematical results obtained. If these results do not give a satisfactory solution to the real problem, then we repeat the same circle again one or more times.

From the time that Pollak presented this scheme in ICME-3 (Karlsruhe, 1976) until nowadays much effort has been placed on analysing in detail the process of MM (Barry & Davies 1996, Edwards & Hauson 1996, Blomhψj & Jensen 2003, Greefrath 2007, Blum & Leib 2007, Voskoglou 1994, 2007, etc.). A brief but comprehensive account of the different models used for the description of the MM process can be found in (Haines & Crouch, 2010).

As a result of all the previously mentioned research it is nowadays more or less acceptable that the main steps of the MM involve:

- **S_1: Analysis** of the problem, i.e. understanding the statement and recognizing the restrictions and requirements of the real system.
- **S_2: Mathematization,** i.e. formulation of the problem in a way that it will be ready for mathematical treatment and construction of the model.
- **S_3: Solution** of the model.
- **S_4: Validation** (control) of the model, which is usually achieved by reproducing, through the model, the behavior of the real system under the conditions existing before the solution of the model and by comparing it to the existing, from the previous "history" of the corresponding real system, data.

• **S_5: Interpretation** of the final mathematical results and implementation of them to the real system, in order to give the "answer" to the real world problem.

Some authors consider further steps in the MM process; e.g. some of them divide mathematization to the distinct steps of formulation of the real problem and construction of the model, others divide the validation to the steps of interpretation and evaluation of the model, others add the step of refining the model, etc. (Haines & Crouch, 2010). All these minor variations, however, do not change the general view that we nowadays have about the circle of MM.

Mathematization possesses the greatest gravity among the steps of the MM process, since it involves a deep abstracting process, which is not always easy to be achieved by a non-expert. As Crouch and Haines (Crouch & Haines, 2004) report, however, it is sometimes the transition from the solution of the model to the real world (validation and interpretation of the model) that presents difficulties for students too. The students' difficulties in solving the following problem when I was teaching, some time ago, the derivatives in a first term university course, illustrates this case.

Problem: We want to construct a channel to run water by folding the two edges of a rectangle metallic leaf having sides of length *20 cm* and *32 cm*, in such a way that they will be perpendicular to the other parts of the leaf. Assuming that the flow of water is constant, how we can run the maximal possible quantity of the water through the channel?

Folding the two edges of the metallic leaf by length x across its longer side the vertical cut of the constructed channel forms a rectangle with sides x and 32-2x (Figure 2).

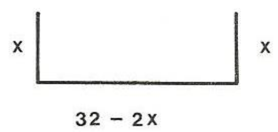

Fig. 2. The vertical cut of the channel

The area of the rectangle, which is equal to $E(x) = x(32-2x) = 32x-2x^2$, has to be maximized. The equation $E'(x) = 32-4x = 0$, where $E'(x)$ denotes the derivative of E(x), gives that x = 8 cm. But $E''(x) = -4 < 0$, therefore $E(8) = 128\ cm^2$ is the maximal possible quantity of water to run through the channel. A number of students, however, folded the edges of the other side of the leaf and they found $E(x) = x(20-2x) = 20x-2x^2$. In this case the equation $E'(x) = 0$ gives that x = 5 cm, while $E(5) = 50\ cm^2$. Their solution was mathematically correct, but many of them failed to realize that it is not acceptable in practice (real world).

MM appears today as a dynamic tool for teaching and learning mathematics, because it connects mathematics with our everyday life giving the possibility to students to understand its usefulness in practice and therefore increasing their interest about mathematics. A special didactic methodology has been developed through the years based on MM, which is usually referred as **application-oriented** teaching of mathematics. But we must be careful. The process of MM could not be considered as a general and therefore applicable to all cases method for teaching mathematics. In fact, such a consideration could lead to far-fetched situations, in which more emphasis is given to the search of the proper applications rather, than to the consolidation of the new mathematical knowledge!

Assessment of MM skills. Quality is a desirable property of all human activities. This makes assessment one of the most important components of those activities. In earlier works the present author has developed several methods for assessing human-machine performance under fuzzy conditions, including the measurement of the uncertainty in fuzzy systems, the use of the Center of Gravity (COG) defuzzification technique, the use of fuzzy or grey numbers, etc. (Voskoglou, 2019a). Here a new model is developed, using soft sets as its basic tools, for assessment with respect to a finite set E of parameters. Such kind of models are very useful when the assessment has qualitative rather than quantitative characteristics, as it happens in this paper with student mathematical modelling skills.

RESULTS AND DISCUSSION

Fuzzy and Soft Sets. The **fuzzy set** theory, introduced by Zadeh in 1965 (Zadeh, 1965), and the connected to it infinite-valued in the interval [0, 1] **fuzzy logic** (Zadeh, 1973) gave to scientists the opportunity to model under conditions of uncertainty that are vague or not precisely defined, thus succeeding to mathematically solve problems whose statements are expressed in the natural language. Through fuzzy logic the fuzzy terminology is translated by algorithmic procedures into numerical values, operations are performed upon those values and the outcomes are returned into natural language statements in a reliable manner.

It is of worth noting that, before the introduction of fuzzy sets, probability used to be the unique tool in hands of the experts for dealing with the existing in real life situations of uncertainty. Probability, however, based on principles of bivalent logic, was proved sufficient for tackling problems of uncertainty connected to randomness only and not those due to imprecision or incomplete information.

It is recalled that a fuzzy set A on the universal set U is defined with the help of its **membership function** $m: U \to [0,1]$ as the set of the ordered pairs

$$A = \{(x, m(x)): x \in U\} \quad (1)$$

The real number m(x) is called the **membership degree** of x in A**.** The greater is m(x), the more x satisfies the characteristic property of A. A crisp subset A of U is a fuzzy set on U with membership function taking the values m(x)=1 if x belongs to A and 0 otherwise. In other words, the concept of fuzzy set is an extension of the concept of the ordinary sets.

Note that there is not any exact rule for defining the membership function of a fuzzy set. The methods used for it are usually empirical or statistical and the definition of the membership function is not unique depending on the personal goals of the observer. The only restriction is to be compatible to the common logic; otherwise the resulting fuzzy set does not give a reliable description of the corresponding real situation. For general facts on fuzzy sets, fuzzy logic and the connected to them uncertainty we refer to the chapters 4-7 of the book (Voskoglou, 2017).

The last 60 years a lot of research has been carried out for generalizing and extending the fuzzy set theory on the purpose of tackling more effectively the existing uncertainty in problems of science, technology and everyday life (Voskoglou, 2019b). One such generalization is the concept of **soft set** proposed in 1999 by Dmtri Molodstov, Professor of the Computing Center of the Russian Academy of Sciences in Moscow, as a new mathematical tool for dealing with the uncertainty in a parametric manner (Molodtsov, 1999).

Let E be a set of parameters, let A be a subset of E and let f be a mapping of A into the set Δ(U) of all subsets of the universal set U. Then the soft set F of U connected to A is defined as the set of the ordered pairs

$$F = \{(e, f€: e \in A\} \quad (2)$$

In other words, a soft set is a paramametrized family of subsets of U. For example, let U= $\{H_1, H_2, H_3\}$ be a set of houses and let E =$\{e_1, e_2, e_3\}$ be the set of the parameters e_1=cheap, e_2=expensive and e_3=beautiful. Let us further assume that H_1, H_2 are the cheap and H_2, H_3 are the beautiful houses. Set A=$\{e_1, e_3\}$, then a mapping f:A $\longrightarrow$ Δ(U) is defined by $f(e_1)=\{H_1, H_2\}$, $f(e_3)=\{H_2, H_3\}$. Therefore, the soft set F of U connected to A and representing the cheap and beautiful houses of U is the set of the ordered pairs

$$F= \{(e_1, \{H_1, H_2\}), (e_3, \{H_2, H_3\})\} \quad (3)$$

A fuzzy set on U with membership function y = m(x) is a soft set on U of the form (f, [0, 1]), where $f(\alpha)=\{x \in U: m(x) \geq \alpha\}$ is the corresponding α – cut of the fuzzy set, for each α in [0. 1]. An important advantage of soft sets is that they pass through the existing difficulty of defining properly the membership function of a fuzzy set. The theory of soft sets has found many and important applications to several sectors of the human activity like decision making, parameter reduction, data clustering and data dealing with incompleteness, etc. (Tripathy & Arun, 2016).

The Assessment Model. Let U be the set of all objects which are under assessment. Consider the set E=$\{e_1,e_2,e_3,e_4,e_5\}$ of the parameters e_1=excellent, e_2=very good, e_3=good, e_4=mediocre and e_5=failed and the mapping f: E $\longrightarrow$ Δ(U) assigning to each parameter of E the subset of U consisting of all elements of U whose performance is described by this parameter. Then the soft set

$$F = \{(e_i, f(e_i), i=1,2,3,4,5\} \quad (4)$$

represents mathematically an assessment of the elements of U in a parametric manner. Note that, for a more detailed assessment the set E could include more than five parameters. The following example illustrates the applicability of this model under real situations.

Example: The coach of a football club wants to assess in a systematic way the following characteristics of his players: D=dribbling, P=passing, F= foot kick (shoot), H=head kick, C=creativity and S=speed.

Set U={D, P, F, H, C, S} and define a mapping f: E $\longrightarrow$ Δ(U) by assigning to each parameter of E and for each player of the club the subset of U consisting of the player's characteristics assessed by this parameter. In this way one can represent each player's profile with the help of a soft set. For example, the soft set

$$F= \{(e_1, \{P, C\}), (e_2, \{F\}), (e_3, \{D\}), (e_4, \{S\}), (e_5, \{H\})\} \quad (8)$$

corresponds to a player with excellent passing and creativity, very good shoot, good dribbling, mediocre speed, but not good head kick.

In an analogous way one can express the general players' performance with respect to each characteristic of U. Consider, for example, dribbling (D) and let V=$\{P_1,P_2,\ldots,P_{20}\}$ be the set of all players of the team. Define a map f: E $\longrightarrow$ Δ(V) assigning to each parameter of E the subset of V consisting of the players whose dribbling was assessed by this parameter. Then, the general players' performance with respect to dribbling is expressed by a soft set of the form (6). We could have, for example, that

$$F= \{(e_1,\{P_1,P_2,P_3\}), (e_2, \{P_4,P_5,\ldots,P_{10}\}), (e_3, \{P_{11}, P_{12},\ldots, P_{15}\}), (e_4, \{P_{16}, P_{17}, P_{18}\}), (e_5, \{ P_{19}, P_{20}\})\} \quad (9)$$

This means that the first three players have excellent dribbling, the next seven very good, the next five good, the next three mediocre and the last two players have no good dribbling.

Assessing Student MM Skills. Let U be the set of all students of a class, say U = $\{P_1, P_2, \ldots, P_n\}$. Assume that the teacher of Mathematics gave to students for solution a series of problems involving MM and assessed their performance in terms of the parameters of the set E. Then the general performance of the class can be represented with the help of a soft set of the form (4). For example, we could have

$$F= \{(e_1,\{P_1,P_2,\ldots, P_5\}), (e_2, \{P_6,P_7,\ldots,P_{12}\}), (e_3, \{P_{13}, P_{14},\ldots, P_{18}\}), (e_4, \{P_{19}, P_{20},\ldots, P_{25}\}), (e_5, \{ P_{26}, P_{27}, \ldots, P_n\})\} \quad (10)$$

This means that the first five students demonstrated excellent performance, the next seven very good, the next six good, the next seven mediocre performance and the rest of the students failed to solve the MM problems successfully.

Another goal of our assessment model is that it enables the representation of the performance of each student in each step of the MM process. In fact, let V = $\{S_1, S_2, S_3, S_4, S_5\}$ be the set of the steps of the MM process described in our Introduction. Consider a particular student of the class and define a map f: E $\longrightarrow$ Δ(V) by assigning to each parameter of E the subset of V consisting of the steps of the MM assessed by this parameter with respect to the chosen student. For example, the soft set

$$F= \{(e_1, \{S_1, S_3\}), (e_2, \{S_5\}), (e_3, \{S4\}), (e_4, \{S2\}), (e_5, \emptyset)\} \quad (11)$$

corresponds to the profile of a student who demonstrated excellent performance at the steps of analysis of the problem and solution of the model, very good performance at the step of interpretation of the mathematical results, good performance at the step of validation of the model and mediocre performance at the step of mathematization (he/she faced difficulties, but he/she finally came through).

CONCLUSION

The discussion performed in this paper leads to the following two conclusions:

- MM is a very useful tool for teaching and learning mathematics, because it gives the opportunity to apply the mathematical theories to practical needs of our everyday life and therefore increases the student interest for mathematics.
- The use of soft sets enables a mathematical representation of a qualitative assessment of human performance with respect to a certain activity in a parametric manner. In particular, the assessment model developed here was applied for the assessment of the general and individual student mathematical modelling skills.

An important subject for future research is the application of the soft set assessment model constructed here to a variety of other human or machine (e.g. computer programs) activities.

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